THE LEFSCHETZ-HOPF THEOREM AND AXIOMS FOR THE LEFSCHETZ NUMBER

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Abstract. The reduced Lefschetz number, that is, $L(\cdot)-1$ where $L(\cdot)$ denotes the Lefschetz number, is proved to be the unique integer-valued function λ on selfmaps of compact polyhedra which is constant on homotopy classes such that (1) $\lambda(fg) = \lambda(gf)$, for $f\colon X \to Y$ and $g\colon Y \to X$; (2) if (f_1, f_2, f_3) is a map of a cofiber sequence into itself, then $\lambda(f_1) = \lambda(f_1) + \lambda(f_3)$; (3) $\lambda(f) = -(\deg(p_1 f e_1) + \cdots + \deg(p_k f e_k))$, where f is a selfmap of a wedge of k circles, e_r is the inclusion of a circle into the rth summand and p_r is the projection onto the rth summand. If $f\colon X \to X$ is a selfmap of a polyhedron and $I(f)$ is the fixed point index of f on all of X, then we show that $I(\cdot)-1$ satisfies the above axioms. This gives a new proof of the Normalization Theorem: If $f\colon X \to X$ is a selfmap of a polyhedron, then $I(f)$ equals the Lefschetz number $L(f)$ of f. This result is equivalent to the Lefschetz-Hopf Theorem: If $f\colon X \to X$ is a selfmap of a finite simplicial complex with a finite number of fixed points, each lying in a maximal simplex, then the Lefschetz number of f is the sum of the indices of all the fixed points of f.

1. Introduction.

Let X be a finite polyhedron and denote by $\widetilde{H}_*(X)$ its reduced homology with rational coefficients. Then the *reduced Euler characteristic* of X, denoted by $\tilde{\chi}(X)$, is defined by

$$\tilde{\chi}(X) = \sum_j (-1)^j \dim \widetilde{H}_j(X).$$

Clearly, $\tilde{\chi}(X)$ is just the Euler characteristic minus one. In 1962, Watts [13] characterized the reduced Euler characteristic as follows: Let ϵ be a function from the set of finite polyhedra with base points to the integers such that (i) $\epsilon(S^0) = 1$, where S^0 is the 0-sphere, and (ii) $\epsilon(X) = \epsilon(A) + \epsilon(X/A)$, where A a subpolyhedron of X. Then $\epsilon(X) = \tilde{\chi}(X)$.

Let $\mathcal{C}$ be the collection of spaces X of the homotopy type of a finite, connected CW-complex. If $X \in \mathcal{C}$, we do not assume that X has a base point except when X is a sphere or a wedge of spheres. It is not assumed that maps between spaces with base points are based. A map $f\colon X \to X$, where $X \in \mathcal{C}$, induces trivial homomorphisms $f_j\colon H_j(X) \to H_j(X)$ of rational homology vector spaces for all $j > \dim X$. The *Lefschetz number* $L(f)$ of f is defined by

$$L(f) = \sum_j (-1)^j Tr\, f_j,$$

1991 *Mathematics Subject Classification.* 55M20.

where Tr denotes the trace. The reduced Lefschetz number $\widetilde{L}$ is given by $\widetilde{L}(f) = L(f) - 1$ or, equivalently, by considering the rational, reduced homology homomorphism induced by f.

Since $\widetilde{L}(id) = \tilde{\chi}(X)$, where $id\colon X \to X$ is the identity map, Watts's Theorem suggests an axiomatization for the reduced Lefschetz number which we state below as Theorem 1.1.

For $k \geq 1$, denote by $\bigvee^k S^n$ the wedge of k copies of the n-sphere $S^n, n \geq 1$. If we write $\bigvee^k S^n$ as $S^n_1 \vee S^n_2 \vee \cdots \vee S^n_k$, where $S^n_j = S^n$, then we have inclusions $e_j\colon S^n_j \to \bigvee^k S^n$ into the j-th summand and projections $p_j\colon \bigvee^k S^n \to S^n_j$ onto the j-th summand, for $j = 1, \ldots, k$. If $f\colon \bigvee^k S^n \to \bigvee^k S^n$ is a map, then $f_j\colon S^n_j \to S^n_j$ denotes the composition $p_j f e_j$. The degree of a map $f\colon S^n \to S^n$ is denoted by $\deg(f)$.

We characterize the reduced Lefschetz number as follows.

Theorem 1.1. *The reduced Lefschetz number $\widetilde{L}$ is the unique function λ from the set of self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the following conditions:*

1. (Homotopy Axiom) If $f, g\colon X \to X$ are homotopic maps, then $\lambda(f) = \lambda(g)$.

2. (Cofibration Axiom) If A is a subpolyhedron of X, $A \to X \to X/A$ is the resulting cofiber sequence and there exists a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow \\ A & \longrightarrow & X & \longrightarrow & X/A, \end{array}$$

then $\lambda(f) = \lambda(f') + \lambda(\bar{f})$.

3. (Commutativity Axiom) If $f\colon X \to Y$ and $g\colon Y \to X$ are maps, then $\lambda(gf) = \lambda(fg)$.

4. (Wedge of Circles Axiom) If $f\colon \bigvee^k S^1 \to \bigvee^k S^1$ is a map, $k \geq 1$, then

$$\lambda(f) = -(\deg(f_1) + \cdots + \deg(f_k)),$$

where $f_j = p_j f e_j$.

In an unpublished dissertation [10], Hoang extended Watts's axioms to characterize the reduced Lefschetz number for basepoint-preserving self-maps of finite polyhedra. His list of axioms is different from, but similar to, those in Theorem 1.1.

One of the classical results of fixed point theory is

Theorem 1.2 (Lefschetz-Hopf). *If $f\colon X \to X$ is a map of a finite polyhedron with a finite set of fixed points, each of which lies in a maximal simplex of X, then $L(f)$ is the sum of the indices of all the fixed points of f.*

The history of this result is described in [3], see also [8, p. 458]. A proof that depends on a delicate argument due to Dold [5] can be found in [2] and, in a more condensed form, in [4]. In an appendix to his dissertation [12], D. McCord outlined a possibly more direct argument, but no details were published. The book of Granas and Dugundji [8, pp. 441 - 450] presents an argument based on classical techniques of Hopf [11]. We use the characterization of the reduced Lefschetz number in Theorem 1.1 to prove the Lefschetz-Hopf theorem in a quite natural manner by showing that the fixed point index satisfies the axioms of Theorem 1.1. That is, we prove

Theorem 1.3 (Normalization Property). *If $f\colon X \to X$ is any map of a finite polyhedron, then $L(f) = i(X, f, X)$, the fixed point index of f on all of X.*

The Lefschetz-Hopf Theorem follows from the Normalization Property by the Additivity Property of the fixed point index. In fact these two statements are equivalent. The Hopf Construction [2, p. 117] implies that a map f from a finite polyhedron to itself is homotopic to a map that satisfies the hypotheses of the Lefschetz-Hopf theorem. Thus the Homotopy and Additivity Properties of the fixed point index imply that the Normalization Property follows from the Lefschetz-Hopf Theorem.

2. Lefschetz numbers and exact sequences.

In this section, all vector spaces are over a fixed field F, which will not be mentioned, and are finite dimensional. A graded vector space $V = \{V_n\}$ will always have the following properties: (1) each V_n is finite dimensional and (2) $V_n = 0$ for $n < 0$ and for $n > N$, for some non-negative integer N. A *map* $f\colon V \to W$ of graded vector spaces $V = \{V_n\}$ and $W = \{W_n\}$ is a sequence of linear transformations $f_n\colon V_n \to W_n$. For a map $f\colon V \to V$, the *Lefschetz number* is defined by

$$L(f) = \sum_n (-1)^n Tr\, f_n.$$

The proof of the following lemma is straightforward, and hence omitted.

Lemma 2.1. *Given a map of short exact sequences of vector spaces*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0,
\end{array}$$

then $Tr\, g = Tr\, f + Tr\, h$. $\square$

Theorem 2.2. *Let A, B and C be graded vector spaces with maps $\alpha\colon A \to B, \beta\colon B \to C$ and selfmaps $f\colon A \to A, g\colon B \to B$ and $h\colon C \to C$. If for every n, there is a linear transformation $\partial_n\colon C_n \to A_{n-1}$ such that the following diagram is commutative and has exact rows:*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & A_N & \xrightarrow{\alpha_N} & B_N & \xrightarrow{\beta_N} & C_N & \xrightarrow{\partial_N} & A_{N-1} & \xrightarrow{\alpha_{N-1}} & \cdots \\
 & & {\scriptstyle f_N}\big\downarrow & & {\scriptstyle g_N}\big\downarrow & & {\scriptstyle h_N}\big\downarrow & & {\scriptstyle f_{N-1}}\big\downarrow & & \\
0 & \longrightarrow & A_N & \xrightarrow{\alpha_N} & B_N & \xrightarrow{\beta_N} & C_N & \xrightarrow{\partial_N} & A_{N-1} & \xrightarrow{\alpha_{N-1}} & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_1} & A_0 & \xrightarrow{\alpha_0} & B_0 & \xrightarrow{\beta_0} & C_0 & \longrightarrow & 0 \\
 & & {\scriptstyle f_0}\big\downarrow & & {\scriptstyle g_0}\big\downarrow & & {\scriptstyle h_0}\big\downarrow & & \\
\cdots & \xrightarrow{\partial_1} & A_0 & \xrightarrow{\alpha_0} & B_0 & \xrightarrow{\beta_0} & C_0 & \longrightarrow & 0,
\end{array}$$

then

$$L(g) = L(f) + L(h).$$

Proof. Let Im denote the image of a linear transformation and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Im\,\beta_n \arrow[r] \arrow[d, "h_n|Im\,\beta_n"'] & C_n \arrow[r] \arrow[d, "h_n"'] & Im\,\partial_n \arrow[r] \arrow[d, "f_{n-1}|Im\,\partial_n"'] & 0 \\
0 \arrow[r] & Im\,\beta_n \arrow[r] & C_n \arrow[r] & Im\,\partial_n \arrow[r] & 0.
\end{tikzcd}$$

By Lemma 2.1, $Tr(h_n) = Tr(h_n|Im\,\beta_n) + Tr(f_{n-1}|Im\,\partial_n)$. Similarly, the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Im\,\partial_n \arrow[r] \arrow[d, "f_{n-1}|Im\,\partial_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"'] & Im\,\alpha_{n-1} \arrow[r] \arrow[d, "g_{n-1}|Im\,\alpha_{n-1}"'] & 0 \\
0 \arrow[r] & Im\,\partial_n \arrow[r] & A_{n-1} \arrow[r] & Im\,\alpha_{n-1} \arrow[r] & 0
\end{tikzcd}$$

yields $Tr(f_{n-1}|Im\,\partial_n) = Tr(f_{n-1}) - Tr(g_{n-1}|Im\,\alpha_{n-1})$. Therefore

$$Tr(h_n) = Tr(h_n|Im\,\beta_n) + Tr(f_{n-1}) - Tr(g_{n-1}|Im\,\alpha_{n-1}).$$

Now consider

$$\begin{tikzcd}
0 \arrow[r] & Im\,\alpha_{n-1} \arrow[r] \arrow[d, "g_{n-1}|Im\,\alpha_{n-1}"'] & B_{n-1} \arrow[r] \arrow[d, "g_{n-1}"'] & Im\,\beta_{n-1} \arrow[r] \arrow[d, "h_{n-1}|Im\,\beta_{n-1}"'] & 0 \\
0 \arrow[r] & Im\,\alpha_{n-1} \arrow[r] & B_{n-1} \arrow[r] & Im\,\beta_{n-1} \arrow[r] & 0,
\end{tikzcd}$$

so $Tr(g_{n-1}|Im\,\alpha_{n-1}) = Tr(g_{n-1}) - Tr(h_{n-1}|Im\,\beta_{n-1})$. Putting this all together, we obtain

$$Tr(h_n) = Tr(h_n|Im\,\beta_n) + Tr(f_{n-1}) - Tr(g_{n-1}) + Tr(h_{n-1}|Im\,\beta_{n-1}).$$

We next look at the left end of the original diagram and get

$$0 = Tr(h_{N+1}) = Tr(f_N) - Tr(g_N) + Tr(h_N|Im\,\beta_N)$$

and at the right end which gives

$$Tr(h_1) = Tr(h_1|Im\,\beta_1) + Tr(f_0) - Tr(g_0) + Tr(h_0).$$

A simple calculation now yields

$$\begin{aligned}
\sum_{n=0}^{N}(-1)^n Tr(h_n) = \sum_{n=0}^{N+1} &= (-1)^n(Tr(h_n|Im\,\beta_n) + Tr(f_{n-1}) - Tr(g_{n-1}) \\
&\qquad + Tr(h_{n-1}|Im\,\beta_{n-1})) \\
&= -\sum_{n=0}^{N}(-1)^n Tr(f_n) + \sum_{n=0}^{N} = (-1)^n Tr(g_n).
\end{aligned}$$

Therefore $L(h) = -L(f) + L(g)$. $\square$

We next give some simple consequences of Theorem 2.2.

If $f\colon (X,A) \to (X,A)$ is a selfmap of a pair, where $X, A \in \mathcal{C}$, then f determines $f_X\colon X \to X$ and $f_A\colon A \to A$. The map f induces homomorphisms $f_j\colon H_j(X,A) \to H_j(X,A)$ of relative homology with coefficients in F. The *relative Lefschetz number* $L(f;X,A)$ is defined by

$$L(f;X,A) = \sum_j (-1)^j Tr f_j.$$

Applying Theorem 2.2 to the homology exact sequence of the pair (X,A), we obtain

Corollary 2.3. *If $f\colon (X,A) \to (X,A)$ is a map of pairs, where $X, A \in \mathcal{C}$, then*

$$L(f; X, A) = L(f_X) - L(f_A).$$

This result was obtained by Bowszyc [1].

Corollary 2.4. *Suppose $X = P \cup Q$ where $X, P, Q \in \mathcal{C}$ and $(X; P, Q)$ is an proper triad* [6, p. 34]*. If $f\colon X \to X$ is a map such that $f(P) \subseteq P$ and $f(Q) \subseteq Q$ then, for f_P, f_Q and $f_{P\cap Q}$ the restrictions of f to P, Q and $P \cap Q$ respectively, we have*

$$L(f) = L(f_P) + L(f_Q) - L(f_{P\cap Q}).$$

Proof. The map f and its restrictions induce a map of the Mayer-Vietoris homology sequence [6, p. 39] to itself so the result follows from Theorem 2.2. □

A similar result was obtained by Ferrario [7, Theorem 3.2.1].

Our final consequence of Theorem 2.2 will be used in the characterization of the reduced Lefschetz number.

Corollary 2.5. *If A is a subpolyhedron of X, $A \to X \to X/A$ is the resulting cofiber sequence of spaces in $\mathcal{C}$ and there exists a commutative diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f'"'] & X \arrow[r] \arrow[d, "f"'] & X/A \arrow[d, "\bar{f}"'] \\
A \arrow[r] & X \arrow[r] & X/A,
\end{tikzcd}$$

then

$$L(f) = L(f') + L(\bar{f}) - 1.$$

Proof. We apply Theorem 2.2 to the homology cofiber sequence. The 'minus one' on the right hand side arises because that sequence ends with

$$\to H_0(A) \to H_0(X) \to \tilde{H}_0(X/A) \to 0. \qquad \square$$

3. Characterization of the Lefschetz number.

Throughout this section, all spaces are assumed to lie in $\mathcal{C}$.

We let λ be a function from the set of self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the Homotopy Axiom, Cofibration Axiom, Commutativity Axiom and Wedge of Circles Axiom of Theorem 1.1 as stated in the Introduction.

We draw a few simple consequences of these axioms. From the Commutativity Axiom, we obtain

Lemma 3.1. *If $f\colon X \to X$ is a map and $h\colon X \to Y$ is a homotopy equivalence with homotopy inverse $k\colon Y \to X$, then $\lambda(f) = \lambda(hfk)$.* □

Lemma 3.2. *If $f\colon X \to X$ is homotopic to a constant map, then $\lambda(f) = 0$.*

Proof. Let $*$ be a one-point space and $*\colon * \to *$ the unique map. From the map of cofiber sequences

$$\begin{array}{ccccc} * & \longrightarrow & * & \longrightarrow & * \\ {\scriptstyle *}\downarrow & & {\scriptstyle *}\downarrow & & {\scriptstyle *}\downarrow \\ * & \longrightarrow & * & \longrightarrow & * \end{array}$$

and the Cofibration Axiom, we have $\lambda(*) = \lambda(*) + \lambda(*)$, and therefore $\lambda(*) = 0$. Write any constant map $c\colon X \to X$ as $c(x) = *$ for some $* \in X$, let $e\colon * \to X$ be inclusion and $p\colon X \to *$ projection. Then $c = ep$ and $pe = *$, and so $\lambda(c) = 0$ by the Commutativity Axiom. The lemma follows from the Homotopy Axiom. □

If X is a based space with base point $*$, i.e., a sphere or wedge of spheres, then the cone and suspension of X are defined by $CX = X \times I/(X \times 1 \cup * \times I)$ and $\Sigma X = CX/(X \times 0)$, respectively.

Lemma 3.3. *If X is a based space, $f\colon X \to X$ is a based map and $\Sigma f\colon \Sigma X \to \Sigma X$ is the suspension of f, then $\lambda(\Sigma f) = -\lambda(f)$.*

Proof. Consider the maps of cofiber sequences

$$\begin{array}{ccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle Cf}\downarrow & & {\scriptstyle \Sigma f}\downarrow \\ X & \longrightarrow & CX & \longrightarrow & \Sigma X. \end{array}$$

Since CX is contractible, Cf is homotopic to a constant map. Therefore, by Lemma 3.2 and the Cofibration Axiom,

$$0 = \lambda(Cf) = \lambda(\Sigma f) + \lambda(f). \qquad \square$$

Lemma 3.4. *For any $k \geq 1$ and $n \geq 1$, if $f\colon \bigvee^k S^n \to \bigvee^k S^n$ is a map, then*

$$\lambda(f) = (-1)^n(\deg(f_1) + \cdots + \deg(f_k)),$$

where $e_r\colon S^n \to \bigvee^k S^n$ and $p_r\colon \bigvee^k S^n \to S^n$ for $r = 1, \ldots, k$ are the inclusions and projections, respectively, and $f_r = p_r f e_r$.

Proof. The proof is by induction on the dimension n of the spheres. The case $n = 1$ is the Wedge of Circles Axiom. If $n \geq 2$, then the map $f\colon \bigvee^k S^n \to \bigvee^k S^n$ is homotopic to a based map $f'\colon \bigvee^k S^n \to \bigvee^k S^n$. Then f' is homotopic to Σg, for some map $g\colon \bigvee^k S^{n-1} \to \bigvee^k S^{n-1}$. Note that if $g_j\colon S_j^{n-1} \to S_j^{n-1}$, then Σg_j is homotopic to $f_j\colon S_j^n \to S_j^n$. Therefore by Lemma 3.3 and the induction hypothesis,

$$\begin{aligned} \lambda(f) = \lambda(f') = -\lambda(g) &= -(-1)^{n-1}((\deg(g_1) + \cdots + \deg(g_k)) \\ &= (-1)^n(\deg(f_1) + \cdots + \deg(f_k)). \quad \square \end{aligned}$$

Proof of Theorem 1.1.

Since $\tilde{L}(f) = L(f) - 1$, Corollary 2.5 implies that $\tilde{L}$ satisfies the Cofibration Axiom. We next show that $\tilde{L}$ satisfies the Wedge of Circles Axiom. There is an isomorphism $\theta\colon \bigoplus^k H_1(S^1) \to H_1(\bigvee^k S^1)$ defined by $\theta(x_1, \ldots, x_k) = e_{1*}(x_1) + \cdots + e_{k*}(x_k)$, where $x_i \in H_1(S^1)$. The inverse $\theta^{-1}\colon H_1(\bigvee^k S^1) \to \bigoplus^k H_1(S^1)$ is given by $\theta^{-1}(y) = (p_{1*}(y), \ldots, p_{k*}(y))$. If $u \in H_1(S^1)$ is a generator, then a basis for $H_1(\bigvee^k S^1)$ is $e_{1*}(u), \ldots, e_{k*}(u)$. By calculating the trace of $f_*\colon H_1(\bigvee^k S^1) \to H_1(\bigvee^k S^1)$ with respect to this basis, we obtain $\tilde{L}(f) = -(\deg(f_1) + \cdots + \deg(f_k))$. The remaining axioms are obviously satisfied by $\tilde{L}$. Thus $\tilde{L}$ satisfies the axioms of Theorem 1.1.

Now suppose λ is a function from the self-maps of spaces in $\mathcal{C}$ to the integers that satisfies the axioms. We regard X as a connected, finite CW-complex and proceed by induction on the dimension of X. If X is 1-dimensional, then it is the homotopy type of a wedge of circles. By Lemma 3.1, we can regard f as a self-map of $\bigvee^k S^1$, and so the Wedge of Circles Axiom gives

$$\lambda(f) = -(\deg(f_1) + \cdots + \deg(f_k)) = \tilde{L}(f).$$

Now suppose that X is n-dimensional and let X^{n-1} denote the $(n-1)$-skeleton of X. Then f is homotopic to a cellular map $g\colon X \to X$ by the Cellular Approximation Theorem [9, Theorem 4.8, p. 349]. Thus $g(X^{n-1}) \subseteq X^{n-1}$, and so we have a commutative diagram

$$\begin{array}{ccccc}
X^{n-1} & \longrightarrow & X & \longrightarrow & X/X^{n-1} = \bigvee^k S^n \\
{\scriptstyle g'}\big\downarrow & & {\scriptstyle g}\big\downarrow & & {\scriptstyle \bar{g}}\big\downarrow \\
X^{n-1} & \longrightarrow & X & \longrightarrow & X/X^{n-1} = \bigvee^k S^n.
\end{array}$$

Then, by the Cofibration Axiom, $\lambda(g) = \lambda(g') + \lambda(\bar{g})$. Lemma 3.4 implies that $\lambda(\bar{g}) = \tilde{L}(\bar{g})$ so, applying the induction hypothesis to g', we have $\lambda(g) = \tilde{L}(g') + \tilde{L}(\bar{g})$. Since we have seen that the reduced Lefschetz number satisfies the Cofibration Axiom, we conclude that $\lambda(g) = \tilde{L}(g)$. By the Homotopy Axiom, $\lambda(f) = \tilde{L}(f)$. $\square$

4. The Normalization Property.

Let X be a finite polyhedron and $f\colon X \to X$ a map. Denote by $I(f)$ the fixed point index of f on all of X, that is, $I(f) = i(X, f, X)$ in the notation of [2] and let $\tilde{I}(f) = I(f) - 1$.

In this section we prove Theorem 1.3 by showing that, with rational coefficients, $I(f) = L(f)$.

Proof of Theorem 1.3.

We will prove that $\tilde{I}$ satisfies the axioms and therefore, by Theorem 1.1, $\tilde{I}(f) = \tilde{L}(f)$. The Homotopy and Commutativity Axioms are well-known properties of the fixed point index (see [2, pp. 59 and 62]).

To show that $\tilde{I}$ satisfies the Cofibration Axiom, it suffices to consider A a subpolyhedron of X and $f(A) \subseteq A$. Let $f'\colon A \to A$ denote the restriction of f and $\bar{f}\colon X/A \to X/A$ the map induced on quotient spaces. Let $r\colon U \to A$ be a deformation retraction of a neighborhood of A in X onto A and let L be a subpolyhedron

of a barycentric subdivision of X such that $A \subseteq int\, L \subseteq L \subseteq U$. By the Homotopy Extension Theorem there is a homotopy $H: X \times I \to X$ such that $H(x, 0) = f(x)$ for all $x \in X$, $H(a, t) = f(a)$ for all $a \in A$ and $H(x, 1) = fr(x)$ for all $x \in L$. If we set $g(x) = H(x, 1)$ then, since there are no fixed points of g on $L - A$, the Additivity Property implies that

$$I(g) = i(X, g, int\, L) + i(X, g, X - L). \tag{4.1}$$

We discuss each summand of (4.1) separately. We begin with $i(X, g, int\, L)$. Since $g(L) \subseteq A \subseteq L$, it follows from the definition of the index ([2, p. 56]) that $i(X, g, int\, L) = i(L, g, int\, L)$. Moreover, $i(L, g, int\, L) = i(L, g, L)$ since there are no fixed points on $L - int\, L$ (the Excision Property of the index). Let $e: A \to L$ be inclusion then, by the Commutativity Property [2, p. 62] we have

$$i(L, g, L) = i(L, eg, L) = i(A, ge, A) = I(f')$$

because $f(a) = g(a)$ for all $a \in A$.

Next we consider the summand $i(X, g, X - L)$ of (4.1). Let $\pi: X \to X/A$ be the quotient map, set $\pi(A) = *$ and note that $\pi^{-1}(*) = A$. If $\bar{g}: X/A \to X/A$ is induced by g, the restriction of $\bar{g}$ to the neighborhood $\pi(int\, L)$ of $*$ in X/A is constant, so $i(X/A, \bar{g}, \pi(int\, L)) = 1$. If we denote the set of fixed points of $\bar{g}$ with $*$ deleted by $Fix_*\bar{g}$, then $Fix_*\bar{g}$ is in the open subset $X/A - \pi(L)$ of X/A. Let W be an open subset of X/A such that $Fix_*\bar{g} \subseteq W \subseteq X/A - \pi(L)$ with the property $\bar{g}(W) \cap \pi(L) = \emptyset$. By the Additivity Property we have

$$I(\bar{g}) = i(X/A, \bar{g}, \pi(int\, L)) + i(X/A, \bar{g}, W) = 1 + i(X/A, \bar{g}, W).$$

Now, identifying $X - L$ with the corresponding subset $\pi(X - L)$ of X/A and identifying the restrictions of $\bar{g}$ and g to those subsets, we have $i(X/A, \bar{g}, W) = i(X, g, \pi^{-1}(W))$. The Excision Property of the index implies that $i(X, g, \pi^{-1}(W)) = i(X, g, X - L)$. Thus we have determined the second summand of (4.1): $i(X, g, X - L) = I(\bar{g}) - 1$.

Therefore from (4.1) we obtain $I(g) = I(f') + I(\bar{g}) - 1$. The Homotopy Property then tells us that

$$I(f) = I(f') + I(\bar{f}) - 1$$

since f is homotopic to g and $\bar{f}$ is homotopic to $\bar{g}$. We conclude that $\tilde{I}$ satisfies the Cofibration Axiom.

It remains to verify the Wedge of Circles Axiom. Let $X = \bigvee^k S^1 = S^1_1 \vee \cdots \vee S^1_k$ be a wedge of circles with basepoint $*$ and $f: X \to X$ a map. We first verify the axiom in the case $k = 1$. We have $f: S^1 \to S^1$ and we denote its degree by $deg(f) = d$. We regard $S^1 \subseteq \mathbb{C}$, the complex numbers. Then f is homotopic to g_d, where $g_d(z) = z^d$ has $|d - 1|$ fixed points for $d \neq 1$. The fixed point index of g_d in a neighborhood of a fixed point that contains no other fixed point of g_d is -1 if $d \geq 2$ and is 1 if $d \leq 0$. Since g_1 is homotopic to a map without fixed points, we see that $I(g_d) = -d + 1$ for all integers d. We have shown that $I(f) = -deg(f) + 1$.

Now suppose $k \geq 2$. If $f(*) = *$ then, by the Homotopy Extension Theorem, f is homotopic to a map which does not fix $*$. Thus we may assume, without loss of generality, that $f(*) \in S^1_1 - \{*\}$. Let V be a neighborhood of $f(*)$ in $S^1_1 - \{*\}$ such that there exists a neighborhood U of $*$ in X disjoint from V with $f(\bar{U}) \subseteq V$.

Since $\bar{U}$ contains no fixed point of f and the open subsets $S^1_j - \bar{U}$ of X are disjoint, the Additivity Property implies

$$I(f) = i(X, f, S^1_1 - \bar{U}) + \sum_{j=2}^{k} i(X, f, S^1_j - \bar{U}). \tag{4.2}$$

The Additivity Property also implies that

$$I(f_j) = i(S^1_j, f_j, S^1_j - \bar{U}) + i(S^1_j, f_j, S^1_j \cap U). \tag{4.3}$$

There is a neighborhood W_j of $(Fix\, f) \cap S^1_j$ in S^1_j such that $f(\overline{W}_j) \subseteq S^1_j$. Thus $f_j(x) = f(x)$ for $x \in W_j$ and therefore, by the Excision Property,

$$i(S^1_j, f_j, S^1_j - \overline{U}) = i(S^1_j, f_j, W_j) = i(X, f, W_j) = i(X, f, S^1_j - \overline{U}). \tag{4.4}$$

Since $f(\overline{U}) \subseteq S^1_1$, then $f_1(x) = f(x)$ for all $x \in \overline{U} \cap S^1_1$. There are no fixed points of f in $\overline{U}$, so $i(S^1_1, f_1, S^1_1 \cap U) = 0$ and thus $I(f_1) = i(X, f, S^1_1 - \overline{U})$ by (4.3) and (4.4).

For $j \geq 2$, the fact that $f_j(U) = *$ gives us $i(S^1_j, f_j, S^1_j \cap U) = 1$ so $I(f_j) = i(X, f, S^1_j - \overline{U}) + 1$ by (4.3) and (4.4). Since $f_j \colon S^1_j \to S^1_j$, the $k = 1$ case of the argument tells us that $I(f_j) = -deg(f_j) + 1$ for $j = 1, 2, \ldots k$. In particular, $i(X, f, S^1_1 - \overline{U}) = -deg(f_1) + 1$ whereas, for $j \geq 2$, we have $i(X, f, S^1_j - \overline{U}) = -deg(f_j)$. Therefore, by (4.2),

$$I(f) = i(X, f, S^1_1 - \overline{U}) + \sum_{j=2}^{k} i(X, f, S^1_j - \overline{U}) = -\sum_{j=1}^{k} deg(f_j) + 1.$$

This completes the proof of Theorem 1.3. □

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